



## Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at <http://about.jstor.org/participate-jstor/individuals/early-journal-content>.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact [support@jstor.org](mailto:support@jstor.org).

# AN EXTENSION OF THE STURM-LIOUVILLE EXPANSION.

BY CHESTER CLAREMONT CAMP.

## Introduction.

In 1836–7 C. Sturm's formal development of a more or less arbitrary function  $f(x)$  in terms of solutions of the self-adjoint equation

$$\frac{d}{dx} \left( k \frac{du}{dx} \right) + (\lambda g - l)u = 0$$

and the Sturmian boundary conditions

$$\begin{aligned} \alpha u(a) + \alpha' u'(a) &= 0, & \left| \alpha \right| + \left| \alpha' \right| &\neq 0 \\ \beta u(b) + \beta' u'(b) &= 0, & \left| \beta \right| + \left| \beta' \right| &\neq 0 \end{aligned}$$

was considered by J. Liouville,\* who undertook the problem of showing that the series converges and that its value is  $f(x)$ . His work is important although it did not satisfy all the requirements of modern mathematical rigor, but in two remarkable papers A. Kneser† completely settled all the more fundamental questions concerning the development. It remained for Haar‡ several years later to give the solution finality.

In 1908 Birkhoff|| extended the theory not only to equations of the  $n$ th order of the form

$$\frac{d^n u}{dx^n} + p_1 \frac{d^{n-1} u}{dx^{n-1}} + \cdots + p_{n-1} \frac{du}{dx} + (p_n + \lambda g)u = 0,$$

but to systems no longer self-adjoint and conditions no longer Sturmian.

The theory is capable of extension in several directions. Bôcher§ considered a system of two homogeneous linear differential equations of the first order and studied the roots of a solution without regard to boundary conditions. Schlesinger¶ took a system of  $n$  linear equations of the first order with coefficients which are rational in  $x$  and obtained the asymptotic forms for a solution.

The object of this paper is to discuss an extension of a problem recently considered by Professor Hurwitz, namely the simultaneous expansion of two

\* *Liouville's Journal*, Vol. 1 (1836), p. 253; Vol. 2 (1837), p. 16 and p. 418.

† *Math. Ann.*, Vol. 58, p. 81 and Vol. 60, p. 402.

‡ Goettingen dissertation (1909) reprinted in *Math. Ann.*, Vol. 69 (1910), p. 331. Also a second paper, *Math. Ann.*, Vol. 71 (1911), p. 38. See also Mercer, *Phil. Trans.*, Vol. 211 (1911), p. 111.

|| *Trans. Amer. Math. Soc.*, Vol. 9, p. 373.

§ *Trans. Amer. Math. Soc.*, Vol. 3 (1902), p. 196.

¶ *Math. Ann.*, Vol. 63 (1907), p. 277.

functions  $f(x)$ ,  $g(x)$  in terms of solutions of the system

$$\begin{aligned}\frac{du}{dx} &= [\lambda + a(x)]v, \\ \frac{dv}{dx} &= -[\lambda + b(x)]u,\end{aligned}$$

and the boundary conditions

$$\begin{aligned}\alpha u(a) + \beta v(a) &= 0, \\ \alpha' u(b) + \beta' v(b) &= 0.\end{aligned}$$

Although this system is simple, it is not self-adjoint and the functions  $a(x)$ ,  $b(x)$  are any two real functions which possess continuous second derivatives.

The extension I have made is that resulting from the substitution of the more general linearly independent boundary conditions

$$\begin{aligned}\alpha_1 u(a) + \beta_1 v(a) + \gamma_1 u(b) + \delta_1 v(b) &= 0, \\ \alpha_2 u(a) + \beta_2 v(a) + \gamma_2 u(b) + \delta_2 v(b) &= 0,\end{aligned}$$

in which the coefficients are such as to satisfy

$$\begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} = \begin{vmatrix} \gamma_1 & \delta_1 \\ \gamma_2 & \delta_2 \end{vmatrix} \neq 0.$$

I wish to acknowledge my indebtedness to Professor Hurwitz for his constant interest and careful direction as well as for his recent article as a basis for research.

For further references the reader is pointed to the following by Bôcher: *Encyklopædie der Math. Wiss.*, Band II, Heft 4, pp. 437–463; *Proceedings of the Fifth International Congress of Mathematicians*, Vol. I (1912), pp. 163–195.

### Section I. Preliminary Lemmas.

Consider the system

$$\begin{cases} N_1(uv\lambda) \equiv u'(x) - [\lambda + a(x)]v(x) = 0, \\ N_2(uv\lambda) \equiv v'(x) + [\lambda + b(x)]u(x) = 0, \end{cases} \quad (1)$$

with the boundary conditions

$$\begin{cases} U_1(uv) \equiv \alpha_1 u(0) + \beta_1 v(0) + \gamma_1 u(1) + \delta_1 v(1) = 0, \\ U_2(uv) \equiv \alpha_2 u(0) + \beta_2 v(0) + \gamma_2 u(1) + \delta_2 v(1) = 0, \end{cases} \quad (2)$$

in which  $\lambda$  is real or complex;  $a(x)$ ,  $b(x)$  are real functions possessing con-

tinuous second derivatives for  $0 \leq x \leq 1$ ; and the coefficients of  $u, v$ , in (2) are real constants such that the matrix\*

$$\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 \end{vmatrix}$$

is of rank two and

$$(\alpha\beta) = (\gamma\delta) \neq 0 \quad (3)$$

where

$$(\alpha\beta) \equiv \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix}.$$

The solution  $u(x) \equiv v(x) \equiv 0$  is called the *trivial* solution of (1), (2). All other solutions are termed *non-trivial*.

Studying non-trivial solutions of (1), (2) we derive the following lemmas.

LEMMA I: If  $(u_m, v_m)$  and  $(u_n, v_n)$  are two solutions of

$$N_1(u_k v_k \lambda_k) = N_2(u_k v_k \lambda_k) = 0,$$

then

$$\begin{aligned} [u_m(x)v_n(x) - u_n(x)v_m(x)]_{x=0}^x \\ = (\lambda_m - \lambda_n) \int_0^x [u_m(s)u_n(s) + v_m(s)v_n(s)] ds. \end{aligned} \quad (4)$$

By hypothesis

$$\begin{aligned} u'_m(x) - [\lambda_m + a(x)]v_m(x) &= 0, \\ v'_m(x) + [\lambda_m + b(x)]u_m(x) &= 0, \\ u'_n(x) - [\lambda_n + a(x)]v_n(x) &= 0, \\ v'_n(x) + [\lambda_n + b(x)]u_n(x) &= 0. \end{aligned}$$

If we multiply these equations by  $v_n(x)$ ,  $-u_n(x)$ ,  $-v_m(x)$ , and  $u_m(x)$  respectively, add, and integrate from 0 to  $x$ , we obtain the equation (4) required.

LEMMA II: If  $(u, v)$  satisfies  $N_1 = \varphi$ ,  $N_2 = \psi$ , where  $\varphi, \psi$  are functions of  $x$  continuous  $0 \leq x \leq 1$ , and  $(U, V)$  satisfies  $N_1 = N_2 = 0$  for the same value of  $\lambda$ , then

$$[u(x)V(x) - U(x)v(x)]_0^x = \int_0^x [V(s)\varphi(s) - U(s)\psi(s)] ds. \quad (5)$$

By hypothesis

$$\begin{aligned} u'(x) - [\lambda + a(x)]v(x) &= \varphi(x), \\ v'(x) + [\lambda + b(x)]u(x) &= \psi(x), \\ U'(x) - [\lambda + a(x)]V(x) &= 0, \\ V'(x) + [\lambda + b(x)]U(x) &= 0. \end{aligned}$$

---

\* If the matrix is of rank less than two the conditions (2) are not independent. The condition  $(\alpha\beta) = (\gamma\delta)$  is somewhat analogous to the condition of self-adjointness for single equations of order two. The case in which  $(\alpha\beta) = (\gamma\delta) = 0$  can obviously be reduced to the problem considered by Professor W. A. Hurwitz.

Multiplying respectively by  $V(x)$ ,  $-U(x)$ ,  $-v(x)$ ,  $u(x)$ ; adding and integrating as before we get equation (5) above.

LEMMA III: *Two pairs of functions  $u(x)$ ,  $v(x)$ ;  $U(x)$ ,  $V(x)$ ; satisfying  $U_1 = U_2 = 0$ , satisfy also the relation*

$$[u(x)V(x) - v(x)U(x)]_0^1 = 0 \quad (6)$$

when (3) is satisfied.

We are given

$$\begin{aligned} \alpha_1 u(0) + \beta_1 v(0) + \gamma_1 u(1) + \delta_1 v(1) &= 0, \\ \alpha_2 u(0) + \beta_2 v(0) + \gamma_2 u(1) + \delta_2 v(1) &= 0, \\ \alpha_1 U(0) + \beta_1 V(0) + \gamma_1 U(1) + \delta_1 V(1) &= 0, \\ \alpha_2 U(0) + \beta_2 V(0) + \gamma_2 U(1) + \delta_2 V(1) &= 0. \end{aligned}$$

Since  $(\alpha\beta) \neq 0$ ,

$$u(0) = -\frac{1}{(\alpha\beta)} \begin{vmatrix} \gamma_1 u(1) + \delta_1 v(1), & \beta_1 \\ \gamma_2 u(1) + \delta_2 v(1), & \beta_2 \end{vmatrix},$$

and

$$v(0) = -\frac{1}{(\alpha\beta)} \begin{vmatrix} \alpha_1, & \gamma_1 u(1) + \delta_1 v(1) \\ \alpha_2, & \gamma_2 u(1) + \delta_2 v(1) \end{vmatrix},$$

also  $U(0)$ ,  $V(0)$  are similar functions of  $U(1)$ ,  $V(1)$ .

Substituting these values in (6) we get for the condition necessary that it be satisfied

$$(\alpha\beta)^2 = (\delta\beta)(\gamma\alpha) + (\gamma\beta)(\alpha\delta) \equiv (\alpha\beta)(\gamma\delta)$$

which is evidently true by (3).

LEMMA IV: *If  $(u_1, v_1)$ ,  $(u_2, v_2)$  are two linearly independent pairs of functions of  $x$ , continuous  $0 \leq x \leq 1$ , then*

$$G_2 \equiv \begin{vmatrix} (11) & (12) \\ (21) & (22) \end{vmatrix}$$

is real and greater than zero, where

$$(rs) \equiv \int_0^1 [u_r(s)\bar{u}_s(s) + v_r(s)\bar{v}_s(s)] ds$$

and  $\bar{u}(x)$  represents the conjugate of  $u(x)$ .

Obviously  $\bar{G}_2 = G_2$ , i.e.,  $G_2$  is real.

Let

$$\begin{aligned} p(x) &\equiv \begin{vmatrix} (11) & u_1(x) \\ (21) & u_2(x) \end{vmatrix}, \\ q(x) &\equiv \begin{vmatrix} (11) & v_1(x) \\ (21) & v_2(x) \end{vmatrix}, \end{aligned}$$

i.e.,

$$\begin{aligned} p(x) &\equiv c_1 u_1(x) + c_2 u_2(x), \\ q(x) &\equiv c_1 v_1(x) + c_2 v_2(x). \end{aligned}$$

Calculate

$$\int_0^1 [p(x)\bar{u}_k(x) + q(x)\bar{v}_k(x)]dx \equiv \begin{vmatrix} (11) & (1k) \\ (21) & (2k) \end{vmatrix} = \begin{cases} 0, & \text{if } k = 1, \\ G_2, & \text{if } k = 2. \end{cases}$$

Then

$$\begin{aligned} \int_0^1 (|p(x)|^2 + |q(x)|^2)dx &\equiv \int_0^1 [p(x)\bar{p}(x) + q(x)\bar{q}(x)]dx \\ &\equiv \bar{c}_1 \int_0^1 [p(x)\bar{u}_1(x) + q(x)\bar{v}_1(x)]dx + \bar{c}_2 \int_0^1 [p(x)\bar{u}_2(x) + q(x)\bar{v}_2(x)]dx \\ &\equiv \bar{c}_2 G_2 \geq 0. \end{aligned}$$

If  $G_2 = 0$  and  $\bar{c}_2 = c_2 \neq 0$ , then  $p(x) \equiv q(x) \equiv 0$  and  $(u_1, v_1), (u_2, v_2)$  will be linearly dependent.

If  $G_2 = 0$  and  $\bar{c}_2 = 0$ , then since

$$c_2 \equiv (11) \equiv \int_0^1 (|u_1(x)|^2 + |v_1(x)|^2)dx = 0,$$

$u_1 \equiv v_1 \equiv 0$  and they will again be linearly dependent, thus violating the hypothesis.

Hence

$$G_2 > 0 \quad (7)$$

since  $\bar{c}_2 > 0$ , and the lemma is proved.\*

LEMMA V: If (3) is satisfied, then

$$[(\gamma\delta) + (\alpha\beta)]^2 \leq [(\alpha\gamma) + (\beta\delta)]^2 + [(\alpha\delta) + (\gamma\beta)]^2. \quad (8)$$

(8) can easily be shown to be equivalent to

$$(\gamma\delta)^2 + (\alpha\beta)^2 \leq (\alpha\gamma)^2 + (\beta\delta)^2 + (\alpha\delta)^2 + (\gamma\beta)^2. \quad (9)$$

But

$$(\alpha\gamma)^2 + (\beta\delta)^2 \geq 2(\alpha\gamma)(\beta\delta),$$

and

$$(\alpha\delta)^2 + (\beta\gamma)^2 \geq -2(\alpha\delta)(\beta\gamma).$$

Hence the right member of (9) is

$$\begin{aligned} &\geq 2(\alpha\gamma)(\beta\delta) - (\alpha\delta)(\beta\gamma) \\ &\geq 2(\alpha\beta)(\gamma\delta), \text{ or by (3)} \\ &\geq (\alpha\beta)^2 + (\gamma\delta)^2. \end{aligned}$$

Since (9) is true, (8) is also.

LEMMA VI: If (3) holds, it is impossible for the coefficients in (2) to satisfy

$$\begin{aligned} (\alpha\gamma) + (\beta\delta) &= 0, \\ (\alpha\delta) + (\gamma\beta) &= 0. \end{aligned} \quad (10)$$

---

\* The lemma admits of the following generalization: If  $(u_1, v_1), (u_2, v_2), \dots, (u_n, v_n)$  are  $n$  linearly independent pairs of functions of  $x$ , continuous  $0 \leq x \leq 1$ , then

$$G_n \equiv \begin{vmatrix} (11) & (12) & \dots & (1n) \\ (21) & (22) & \dots & (2n) \\ \vdots & \vdots & \ddots & \vdots \\ (n1) & (n2) & \dots & (nn) \end{vmatrix}$$

is real and greater than zero,  $n$  being a positive integer.

By (3)

$$(\alpha\beta) - (\gamma\delta) = 0.$$

If then we assume (10) true, by squaring and adding all these equations, we have

$$\begin{aligned} (\alpha\gamma)^2 + (\beta\delta)^2 + (\alpha\delta)^2 + (\gamma\beta)^2 + (\alpha\beta)^2 + (\gamma\delta)^2 + 2(\alpha\gamma)(\beta\delta) \\ + 2(\alpha\delta)(\gamma\beta) - 2(\alpha\beta)(\gamma\delta) = 0. \end{aligned}$$

The sum of the last three terms of the first member vanishes identically so that if (10) holds, each of the six determinants must vanish. Since this violates (3) the proof is complete.

## Section II. Properties of Solutions of the Homogeneous and Non-homogeneous Systems.

LEMMA I: *A necessary and sufficient condition for the existence of a solution  $(u, v)$  of (1), (2) is that the determinant*

$$D(\lambda) \equiv \begin{vmatrix} U_1(u_1v_1) & U_1(u_2v_2) \\ U_2(u_1v_1) & U_2(u_2v_2) \end{vmatrix} \quad (11)$$

*vanish for some value of  $\lambda$ , where  $(u_1, v_1)$ ,  $(u_2, v_2)$  are solutions of (1) defined by*

$$\begin{cases} u_1(0) = 1, & v_1(0) = 0, \\ u_2(0) = 0, & v_2(0) = 1. \end{cases} \quad (12)$$

By the existence theorem we know that a solution either of (1) or of the corresponding non-homogeneous system

$$N_1 = \varphi, \quad N_2 = \psi,$$

where  $\varphi, \psi$  are functions of  $x$ , continuous  $0 \leq x \leq 1$ , will be entire in  $\lambda$ , provided it is defined by

$$\begin{cases} u(0) = c_1, \\ v(0) = c_2, \end{cases}$$

in which the  $c$ 's are independent of  $\lambda$ . Clearly  $D(\lambda)$  will also be entire in  $\lambda$ .

Any solution of (1) is expressible in the form

$$\begin{cases} u(x) = u(0)u_1(x) + v(0)u_2(x), \\ v(x) = u(0)v_1(x) + v(0)v_2(x). \end{cases} \quad (13)$$

In order that  $u(x), v(x)$  satisfy (2) it is necessary and sufficient that  $u(0), v(0)$  be determined so as to satisfy

$$\begin{cases} U_1(u) \equiv U_1(u_1v_1)u(0) + U_1(u_2v_2)v(0) = 0, \\ U_2(u) \equiv U_2(u_1v_1)u(0) + U_2(u_2v_2)v(0) = 0, \end{cases} \quad (14)$$

or if  $u(x), v(x)$  is to be a non-trivial solution, the determinant of coefficients of  $u(0), v(0)$ , which is  $D(\lambda)$ , must vanish. This is obviously also sufficient.

LEMMA II: *The roots of  $D(\lambda)$  are all real and the solutions of (1), (2) may be taken as real.*

Let  $\lambda = \lambda_k$  be a root of  $D(\lambda)$ . Then  $(u_k, v_k)$  satisfies  $N_1(u_k v_k \lambda_k) = 0$ ,  $N_2(u_k v_k \lambda_k) = 0$  and  $(\bar{u}_k, \bar{v}_k)$  will satisfy the same conditions for  $\lambda = \bar{\lambda}_k$ . Hence by Lemma I, Section I

$$[\bar{u}_k v_k - u_k \bar{v}_k]_0^x = (\bar{\lambda}_k - \lambda_k) \int_0^x (|u_k(s)|^2 + |v_k(s)|^2) ds.$$

Moreover  $(\bar{u}_k, \bar{v}_k)$  satisfies  $U_1(u) = U_2(u) = 0$  so that by Lemma III, Section I

$$(\bar{\lambda}_k - \lambda_k) \int_0^1 (|u_k|^2 + |v_k|^2) dx = 0.$$

If  $\bar{\lambda}_k \neq \lambda_k$ ,  $(u_k, v_k)$  is a trivial solution which we have excluded. Therefore,  $\bar{\lambda}_k = \lambda_k$  and  $\lambda_k$  is real.

Since each equation of the system (1), (2) is linear and  $a(x)$ ,  $b(x)$  are real, any solution of the system if complex may be broken up into real and imaginary parts which will separately satisfy the same system. When  $D(\lambda)$  vanishes the ratio of  $u(0)$  to  $v(0)$  or of  $v(0)$  to  $u(0)$  is determinable\* and by the existence theorem the solution  $u(x)$ ,  $v(x)$  is then uniquely determined except for a constant factor. Hence the coefficients of the real and imaginary parts constitute two solutions which are linearly dependent. It is therefore clear that we may restrict ourselves to real solutions of the system.

A similar discussion will show that we need consider only real solutions of the system

$$N_1 = \varphi, \quad N_2 = \psi, \quad (15)$$

where  $\varphi, \psi$  are real functions of  $x$  and  $\lambda$  is restricted to real values, a restriction which we shall henceforth make.

A value of  $\lambda$  which makes  $D(\lambda)$  vanish is called a *principal parameter value* and the corresponding solution of the homogeneous system (1), (2) is known as a principal solution.

Since  $D(\lambda)$  is an entire function it cannot have more than a finite number of roots in any finite interval of the  $\lambda$ -axis, so that we may designate the roots of  $D(\lambda)$  by  $\lambda_n$ ,  $n = 0, \pm 1, \pm 2, \dots$ .

THEOREM I: *A sufficient condition for the existence of a solution  $(\bar{u}, \bar{v})$  of (15), (2) analytic for all real finite values of  $\lambda$  is that*

$$\int_0^1 \begin{vmatrix} u_n(s) & \varphi(s) \\ v_n(s) & \psi(s) \end{vmatrix} ds = 0 \quad (16)$$

for every  $(u_n, v_n)$  satisfying (1), (2) for  $\lambda = \lambda_n$ ,  $n = 0, \pm 1, \pm 2, \dots$ .

We shall consider in turn three kinds of values of  $\lambda$ :

Case I: Those for which the matrix of (11) is of rank two;

---

\* If the matrix of (11) is of rank zero (Case III below), then this is found by the evaluation of an indeterminate form.

Case II: Those for which the matrix is of rank one (at  $\lambda = \lambda_n$ );

Case III: Those for which it is of rank zero (at  $\lambda = \lambda_k$ ).

Take a solution of (1),

$$\begin{cases} u(x) = c_1 u_1(x) + c_2 u_2(x), \\ v(x) = c_1 v_1(x) + c_2 v_2(x), \end{cases} \quad (17)$$

which is identical with (13), provided  $u(0) = c_1$ ,  $v(0) = c_2$ . This solution will satisfy (2) if (14) is satisfied and will be non-trivial provided  $c_1, c_2$  are not both zero. For Case I at least two of the elements of  $D(\lambda)$  must be different from zero. Without loss of generality assume that  $U_1(u_2 v_2) \neq 0$ , then by continuity it will not vanish for values of  $\lambda$  nearby. Put  $c_1 \equiv -U_1(u_2 v_2)$ ,  $c_2 \equiv U_1(u_1 v_1)$ . Then  $U_1(uv) \equiv 0$ , and  $U_2(uv) \equiv D(\lambda)$ .

Define another solution  $(U, V)$  of (1) by

$$\begin{cases} U(x) \equiv -c_2 u_1(x) + c_1 u_2(x), \\ V(x) \equiv -c_2 v_1(x) + c_1 v_2(x). \end{cases} \quad (18)$$

Then  $(u, v)$ ,  $(U, V)$  will be linearly independent since

$$W \equiv \begin{vmatrix} u(0) & v(0) \\ U(0) & V(0) \end{vmatrix} = \begin{vmatrix} c_1 & c_2 \\ -c_2 & c_1 \end{vmatrix} = c_1^2 + c_2^2 > 0.$$

Consider a solution  $(\tilde{u}, \tilde{v})$  of (15) defined by

$$\begin{cases} \tilde{u}(x) \equiv u_0(x) + b_1 u(x) + b_2 U(x), \\ \tilde{v}(x) \equiv v_0(x) + b_1 v(x) + b_2 V(x), \end{cases} \quad (19)$$

where  $(u_0, v_0)$  is a particular solution of (15) such that

$$u_0(0) = v_0(0) = 0.$$

Then

$$\begin{cases} U_1(\tilde{u}\tilde{v}) = U_1(u_0 v_0) - b_2 W, \\ U_2(\tilde{u}\tilde{v}) = U_2(u_0 v_0) + b_1 U_2(uv) + b_2 U_2(UV), \end{cases} \quad (20)$$

and  $(\tilde{u}, \tilde{v})$  will satisfy (2) if

$$\begin{aligned} b_2 &= \frac{U_1(u_0 v_0)}{W}, \\ b_1 &= \frac{-U_1(u_0 v_0)U_2(UV) - U_2(u_0 v_0)W}{WU_2(uv)}, \end{aligned} \quad (21)$$

$$\begin{aligned} \tilde{u}(x) &= u_0(x) + \frac{U_1(u_0 v_0)U(x)}{W} - \frac{U_1(u_0 v_0)U_2(UV) + U_2(u_0 v_0)W}{WD(\lambda)} u(x), \\ \tilde{v}(x) &= v_0(x) + \frac{U_1(u_0 v_0)V(x)}{W} - \frac{U_1(u_0 v_0)U_2(UV) + U_2(u_0 v_0)W}{WD(\lambda)} v(x), \end{aligned} \quad (22)$$

since  $U_2(uv) = D(\lambda)$ .

Hence if the matrix of (11) is of rank two, (22) determines a solution

$(\tilde{u}, \tilde{v})$  analytic in  $\lambda$  for that range for which  $U_1(u_2v_2) \neq 0$ . For other ranges of values of  $\lambda$  within which  $U_1(u_2v_2)$  vanishes we choose in its place another element of the determinant  $D(\lambda)$  which differs from zero throughout that range. This process determines  $b_1, b_2$  in a different fashion. Clearly since the roots of  $U_1(u_2v_2)$  and of every other element of  $D(\lambda)$  are isolated, each element being entire in  $\lambda$ , the solution  $(\tilde{u}, \tilde{v})$  determined in either way for points of the  $\lambda$ -axis at which  $U_1(u_2v_2) \neq 0$  and some other element, e.g.,  $U_2(u_1v_1) \neq 0$ , will be identical since there is one and only one such solution by the existence theorem.  $\therefore$  The proof is complete for Case I.

Case II: Let us first show that no root of  $D(\lambda)$  is double for this case. Assuming that  $U_1(u_2v_2) \neq 0$  for  $\lambda = \lambda_n$  and defining  $c_1, c_2$  as before, we have by differentiating as to  $\lambda$ , since  $u(x), v(x)$  of (17) are entire in  $\lambda$  and satisfy (1):

$$\begin{cases} N_1(u_\lambda v_\lambda \lambda) = v(x), \\ N_2(u_\lambda v_\lambda \lambda) = -u(x). \end{cases} \quad (23)$$

If  $D(\lambda)$  has a double root  $\lambda = \lambda_n$ , then for this value of  $\lambda$

$$\begin{aligned} U_2(uv) = 0, \quad U_2(u_\lambda v_\lambda) = 0; \quad N_1(uv\lambda) = N_2(uv\lambda) = 0; \\ U_1(uv) = 0, \quad U_1(u_\lambda v_\lambda) = 0. \end{aligned}$$

Whence by Lemmas II, III, Section I

$$\int_0^1 [u(s)]^2 + [v(s)]^2 ds = 0.$$

This is impossible since  $u(0) \equiv -U_1(u_2v_2) \neq 0$ . Therefore  $D(\lambda)$  has no double root.

For values sufficiently near  $\lambda_n$ ,  $D(\lambda) \neq 0$  and the solution  $(\tilde{u}, \tilde{v})$  of (15), (2) analytic in  $\lambda$  is given by (22). And by Lemma II, Section I

$$[u_0(x)v(x) - u(x)v_0(x)]_0^x = \int_0^x [v(s)\varphi(s) - u(s)\psi(s)]ds, \quad (24)$$

$$[u_0(x)V(x) - U(x)v_0(x)]_0^x = \int_0^x [V(s)\varphi(s) - U(s)\psi(s)]ds. \quad (25)$$

The left members vanish at the lower limit since  $u_0(0)$  and  $v_0(0)$  are equal to zero, hence solving we obtain

$$u_0(x) = -\frac{1}{W} \int_0^x \begin{vmatrix} u(x) & U(x) & 0 \\ u(s) & U(s) & \varphi(s) \\ v(s) & V(s) & \psi(s) \end{vmatrix} ds, \quad (26)$$

$$v_0(x) = -\frac{1}{W} \int_0^x \begin{vmatrix} v(x) & V(x) & 0 \\ u(s) & U(s) & \varphi(s) \\ v(s) & V(s) & \psi(s) \end{vmatrix} ds. \quad (27)$$

If  $\lambda \rightarrow \lambda_n$ ,  $\tilde{u}(x)$  approaches the value

$$u_0(x) + \frac{U_1(u_0v_0)}{W} U(x) - \frac{u(x)}{W} \left\{ \frac{\frac{\partial}{\partial \lambda} [U_1(u_0v_0)U_2(UV) + U_2(u_0v_0)W]}{D'(\lambda)} \right\}_{\lambda=\lambda_n} \quad (28)$$

provided only

$$U_1(u_0v_0)U_2(UV) + U_2(u_0v_0)W \Big|_{\lambda=\lambda_n} = 0 \quad (29)$$

since  $D'(\lambda) \neq 0$ .

Again if (29) holds,  $\tilde{v}(x)$  will approach a limit as  $\lambda \rightarrow \lambda_n$ . In such a case  $(\tilde{u}, \tilde{v})$  will be continuous if put equal to the limit approached at  $\lambda = \lambda_n$  and therefore analytic for all  $\lambda$  in the interval.

Since  $W = -U_1(UV)$ , from (26), (27) it follows that (29) is equivalent to

$$\int_0^1 \begin{vmatrix} \gamma_1 u(1) + \delta_1 v(1), & U_1(UV) \\ \gamma_2 u(1) + \delta_2 v(1), & U_2(UV) \end{vmatrix}, \begin{vmatrix} \gamma_1 U(1) + \delta_1 V(1), & U_1(UV) \\ \gamma_2 U(1) + \delta_2 V(1), & U_2(UV) \end{vmatrix}, \begin{vmatrix} 0 \\ 0 \end{vmatrix} \begin{vmatrix} \varphi(s) \\ \psi(s) \end{vmatrix}_{\lambda=\lambda_n} ds = 0. \quad (30)$$

The first element of the large determinant is the same as

$$\begin{vmatrix} \gamma_1 u(1) + \delta_1 v(1), & \gamma_1 U(1) + \delta_1 V(1) \\ \gamma_2 u(1) + \delta_2 v(1), & \gamma_2 U(1) + \delta_2 V(1) \end{vmatrix} - \begin{vmatrix} \alpha_1 u(0) + \beta_1 v(0), & \alpha_1 U(0) + \beta_1 V(0) \\ \alpha_2 u(0) + \beta_2 v(0), & \alpha_2 U(0) + \beta_2 V(0) \end{vmatrix}$$

since for  $\lambda = \lambda_n$ ,  $U_1(uv) = U_2(uv) = 0$ , or simplifying further, this element reduces to  $(\gamma\delta)W - (\alpha\beta)W = 0$ .

Thus a sufficient condition for the existence of a solution  $(\tilde{u}, \tilde{v})$  of (15), (2) analytic for all  $\lambda$  when  $D(\lambda)$  is of rank one is that (16) be satisfied for all  $n$  such that  $D(\lambda_n) = 0$ , and the theorem is proved for Case II.

Case III. When every element of  $D(\lambda)$  vanishes for some  $\lambda = \lambda_k$ , it is obvious that  $D'(\lambda_k)$  also vanishes. Let us show that  $D''(\lambda_k) \neq 0$  for such a case.

By the same reasoning as was used to prove  $D'(\lambda) \neq 0$  for  $\lambda = \lambda_n$  in Case II we show that if  $U_1(u_{1\lambda}v_{1\lambda})$  and  $U_2(u_{1\lambda}v_{1\lambda})$  both vanish at  $\lambda = \lambda_k$ , then

$$\int_0^1 ([u_1(x)]^2 + [v_1(x)]^2) dx = 0,$$

which is impossible since  $u_1(0) = 1$ . Hence  $U_1(u_{1\lambda}v_{1\lambda})$  and  $U_2(u_{1\lambda}v_{1\lambda})$  cannot both vanish at  $\lambda = \lambda_k$ . Similarly  $U_1(u_{2\lambda}v_{2\lambda})$  and  $U_2(u_{2\lambda}v_{2\lambda})$  cannot both vanish there. Without loss of generality we may assume

$$U_1(u_{2\lambda}v_{2\lambda}) \neq 0, \quad \lambda = \lambda_k. \quad (31)$$

Now define a solution of (1) by

$$\begin{cases} u(x) \equiv u_1(x) + \frac{c_1}{c_2} u_2(x), \\ v(x) \equiv v_1(x) + \frac{c_1}{c_2} v_2(x), \end{cases} \quad (32)$$

and choose

$$\frac{c_1}{c_2} \equiv -\frac{U_1(u_1v_1)}{U_1(u_2v_2)}.$$

For values of  $\lambda$  sufficiently near  $\lambda_k$ ,  $U_1(u_2v_2) \neq 0$ , otherwise (31) would be violated by Rolle's Theorem. Hence for all  $\lambda$  in the interval considered

$$U_1(uv) \equiv U_1(u_1v_1) - \frac{U_1(u_1v_1)}{U_1(u_2v_2)} U_1(u_2v_2) \equiv 0.$$

Consequently for all such  $\lambda$ ,  $U_1(u_\lambda v_\lambda) \equiv 0$ .

Again

$$U_2(uv) = U_2(u_1v_1) - \frac{U_1(u_1v_1)}{U_1(u_2v_2)} U_2(u_2v_2) = \frac{-D(\lambda)}{U_1(u_2v_2)}, \quad \lambda \neq \lambda_k.$$

At  $\lambda = \lambda_k$ ,

$$U_2(uv) = -\frac{D'(\lambda)}{U_1(u_{2\lambda}v_{2\lambda})} = 0.$$

And

$$U_2(u_\lambda v_\lambda) = -\frac{D'(\lambda)}{U_1(u_2v_2)} + \frac{D(\lambda)U_1(u_{2\lambda}v_{2\lambda})}{[U_1(u_2v_2)]^2}, \quad \lambda \neq \lambda_k.$$

For  $\lambda = \lambda_k$ ,

$$U_2(u_\lambda v_\lambda) = -\frac{D''(\lambda)}{U_1(u_{2\lambda}v_{2\lambda})} + \frac{D(\lambda)U_1(u_{2\lambda\lambda}v_{2\lambda\lambda}) + D'(\lambda)U_1(u_{2\lambda})}{2U_1(u_2v_2)U_1(u_{2\lambda}v_{2\lambda})}.$$

This vanishes if  $D''(\lambda) = 0$  at  $\lambda = \lambda_k$ . Applying the argument a third time we obtain

$$\int_0^1 (u^2 + v^2) dx = 0,$$

which is absurd since by (32)  $u(0) = 1$ .

$$\therefore D''(\lambda) \neq 0 \text{ at } \lambda = \lambda_k.$$

Since for Case III when  $\lambda = \lambda_k$  every solution of (1) satisfies (2), if we define a solution of (15) by

$$\begin{cases} \tilde{u}(x) \equiv u_0(x) + b_1u_1(x) + b_2u_2(x), \\ \tilde{v}(x) \equiv v_0(x) + b_1v_1(x) + b_2v_2(x), \end{cases} \quad (33)$$

where  $u_0(0) = v_0(0) = 0$  defines a particular solution of (15), then  $U_1(\tilde{u}\tilde{v}) = U_1(u_0v_0)$  and  $U_2(\tilde{u}\tilde{v}) = U_2(u_0v_0)$ . If these vanish  $b_1, b_2$  will be arbitrary. Again by using results similar to (24), (25), (26), (27) we have, since

$$\begin{vmatrix} u_1(0) & v_1(0) \\ u_2(0) & v_2(0) \end{vmatrix} = 1,$$

$$U_1(u_0v_0) \equiv - \int_0^1 \begin{vmatrix} \gamma_1u_1(1) + \delta_1v_1(1), & \gamma_1u_2(1) + \delta_1v_2(1), & 0 \\ u_1(s), & u_2(s), & \varphi(s) \\ v_1(s), & v_2(s), & \psi(s) \end{vmatrix} ds,$$

$$U_2(u_0v_0) \equiv - \int_0^1 \begin{vmatrix} \gamma_2u_1(1) + \delta_2v_1(1), & \gamma_2u_2(1) + \delta_2v_2(1), & 0 \\ u_1(s), & u_2(s), & \varphi(s) \\ v_1(s), & v_2(s), & \psi(s) \end{vmatrix} ds.$$

For  $\lambda = \lambda_k$ ,  $(u_1, v_1)$ ,  $(u_2, v_2)$  are solutions of (1), (2). Hence if we assume

that (16) holds for all solutions  $(u_n, v_n)$  of (1), (2), then  $(\tilde{u}, \tilde{v})$  will satisfy (2) for  $b_1, b_2$  arbitrary.

We wish to choose values for them such as to make the solution  $(\tilde{u}, \tilde{v})$ , now analytic  $\lambda \neq \lambda_k$ , continuous at  $\lambda = \lambda_k$ .

Since every solution of (1) may be expressed as a linear combination of  $u_1, u_2, v_1, v_2$ , we may put (33) in the form

$$\begin{cases} \tilde{u}(x) \equiv u_0(x) + d_1 u(x) + d_2 u_2(x), \\ \tilde{v}(x) \equiv v_0(x) + d_1 v(x) + d_2 v_2(x), \end{cases} \quad (34)$$

in which  $(u, v)$  is defined by (32) and  $c_1/c_2$  has the same value as before.  $(u, v)$  and  $(u_2, v_2)$  are linearly independent since

$$\begin{vmatrix} u(0) & v(0) \\ u_2(0) & v_2(0) \end{vmatrix} = \begin{vmatrix} 1 & c_1/c_2 \\ 0 & 1 \end{vmatrix} \neq 0.$$

As  $U_1(uv) \equiv 0$ , we have

$$U_1(\tilde{u}\tilde{v}) = U_1(u_0v_0) + d_2 U_1(u_2v_2),$$

which will vanish for all  $\lambda$  in the interval considered if

$$d_2 \equiv -\frac{U_1(u_0v_0)}{U_1(u_2v_2)}.$$

Again

$$U_2(\tilde{u}\tilde{v}) = U_2(u_0v_0) + d_1 U_2(uv) + d_2 U_2(u_2v_2) = 0,$$

provided

$$d_1 = -\frac{U_2(u_0v_0)U_1(u_2v_2) + U_1(u_0v_0)U_2(u_2v_2)}{U_1(u_2v_2)U_2(uv)}.$$

For  $\lambda \neq \lambda_k$ ,

$$U_2(uv) = -\frac{D(\lambda)}{U_1(u_2v_2)}$$

and

$$d_1 = \frac{U_2(u_0v_0)U_1(u_2v_2) + U_1(u_0v_0)U_2(u_2v_2)}{D(\lambda)}.$$

Hence for  $\lambda = \lambda_k$  in a sufficiently small interval,  $D(\lambda) \neq 0$  and  $U_1(u_2, v_2) \neq 0$ ,

$$\tilde{u}(x) = u_0(x) + \frac{U_2(u_0v_0)U_1(u_2v_2) + U_1(u_0v_0)U_2(u_2v_2)}{D(\lambda)} u(x) - \frac{U_1(u_0v_0)}{U_1(u_2v_2)} u_2(x). \quad (35)$$

If we let  $\lambda$  approach  $\lambda_k$  we get as the value approached by the right member:

$$u_0(x) - \frac{U_1(u_{0\lambda}v_{0\lambda})}{U_1(u_{2\lambda}v_{2\lambda})} u_2(x) + u(x) \frac{\frac{\partial^2}{\partial \lambda^2} [U_2(u_{0\lambda}v_{0\lambda}) + U_1(u_{0\lambda}v_{0\lambda})U_2(u_{2\lambda}v_{2\lambda})]}{D''(\lambda)}, \quad \lambda = \lambda_k.$$

Evidently  $u(x)$  approaches a limit since

$$\frac{c_1}{c_2} \rightarrow -\frac{U_1(u_{1\lambda}v_{1\lambda})}{U_1(u_{2\lambda}v_{2\lambda})}.$$

Also

$$\frac{\partial}{\partial \lambda} [U_2(u_0 v_0) U_1(u_2 v_2) + U_1(u_0 v_0) U_2(u_2 v_2)] = 0, \quad \lambda = \lambda_k,$$

since each factor of both terms vanishes there. Also  $D'(\lambda_k) = 0$ .

Hence  $\tilde{u}(x)$  approaches a limit as  $\lambda \rightarrow \lambda_k$ , and similarly  $\tilde{v}(x)$  does also. Thus we have a solution  $(\tilde{u}, \tilde{v})$  analytic for all  $\lambda$  in a small enough interval about  $\lambda = \lambda_k$ , which satisfies (2). By extending the reasoning as in Case I the proof is completed.

THEOREM II: If for  $n = 0, \pm 1, \pm 2, \dots$

$$\int_0^1 \begin{vmatrix} u_n(s) & \varphi(s) \\ v_n(s) & \psi(s) \end{vmatrix} ds = 0,$$

then  $\varphi(x) \equiv \psi(x) \equiv 0$ .

The proof will be merely outlined here since it is given in full in a recent article by Professor W. A. Hurwitz.

Let the solution of (15), (2) shown by the previous theorem to exist be represented by

$$\begin{aligned} u(x) &\equiv y_0(x) + \lambda y_1(x) + \lambda^2 y_2(x) + \dots, \\ v(x) &\equiv z_0(x) + \lambda z_1(x) + \lambda^2 z_2(x) + \dots. \end{aligned} \quad (36)$$

Since  $\lambda$  is restricted to real values we have but real functions of  $x$  with which to deal. Putting (36) in (15), (2) and equating coefficients of  $\lambda^k$  we get sets of differential equations and boundary conditions satisfied by  $(y_m, z_m)$ ,  $m = 0, 1, 2, \dots$ . By eliminating  $a(x)$ ,  $b(x)$  from two different sets and using Lemma III, Section I, one gets

$$\int_0^1 (y_m y_n + z_m z_n) dx = (\int_0^1 y_{m-1} y_{n+1} + z_{m-1} z_{n+1})$$

where

$$W_r \equiv \int_0^1 (y_r y_0 + z_r z_0) dx.$$

Then from the inequality

$$\begin{aligned} &\left| \frac{y_{m+1}(x)}{y_{m-1}(x)} - \frac{y_{m+1}(\xi)}{y_{m-1}(\xi)} \right|^2 + \left| \frac{y_{m+1}(x)}{y_{m-1}(x)} - \frac{z_{m+1}(\xi)}{z_{m-1}(\xi)} \right|^2 \\ &\quad + \left| \frac{z_{m+1}(x)}{z_{m-1}(x)} - \frac{y_{m+1}(\xi)}{y_{m-1}(\xi)} \right|^2 + \left| \frac{z_{m+1}(x)}{z_{m-1}(x)} - \frac{z_{m+1}(\xi)}{z_{m-1}(\xi)} \right|^2 \geq 0, \end{aligned} \quad (37)$$

for  $x$  and  $\xi$  in the interval from zero to one, by integrating as to  $x$  and  $\xi$  successively from 0 to 1 and simplifying, we have

$$W_{2m-2} W_{2m+2} - W_{2m}^2 \geq 0. \quad (38)$$

Next comes the lemma that some  $W_{2k} = 0$ . Assume no  $W_{2k} = 0$ . Multiplying the series of (36) by  $y_0(x)$ ,  $z_0(x)$  respectively, adding and integrating

we obtain the uniformly convergent series

$$W_0 + \lambda W_1 + \lambda^2 W_2 + \dots \quad (39)$$

This converges absolutely as also does

$$|W_0| + \lambda^2 |W_2| + \lambda^4 |W_4| + \dots \quad (40)$$

Using (38), i.e.

$$\frac{W_0}{W_2} \geq \frac{W_2}{W_4} \geq \frac{W_4}{W_6} \geq \dots,$$

for  $\lambda = \sqrt{\left| \frac{W_0}{W_2} \right|}$  in this we get an absurdity:

$|W_0| + |W_0| + |W_0| + \dots$  convergent. Thus the lemma is proved, i.e.

$$\int_0^1 (y_k^2 + z_k^2) dx = 0, \quad \text{or} \quad y_k(x) \equiv z_k(x) \equiv 0. \quad (41)$$

Thence from the differential equations

$$y_{k-1}(x) \equiv y_{k-2}(x) \equiv \dots \equiv y_0(x) \equiv 0,$$

and similarly for  $z$  and also  $\psi(x) \equiv \varphi(x) \equiv 0$ . Q.e.d.

### Section III. Asymptotic Formulæ.

THEOREM III: For  $|\lambda|$  large a solution of (1) defined by  $u(0) = \alpha$ ,  $v(0) = \beta$  takes the form

$$\begin{cases} u(x) = \alpha \cos \xi + \beta \sin \xi + O\left(\frac{1}{\lambda}\right), \\ v(x) = \beta \cos \xi - \alpha \sin \xi + O\left(\frac{1}{\lambda}\right), \end{cases} \quad (42)$$

and its partial derivatives, the form

$$\begin{cases} u_\lambda(x) = \beta x \cos \xi - \alpha x \sin \xi + O\left(\frac{1}{\lambda}\right), \\ v_\lambda(x) = -\alpha x \cos \xi - \beta x \sin \xi + O\left(\frac{1}{\lambda}\right), \end{cases} \quad (43)$$

where

$$\xi \equiv \lambda x + \frac{1}{2} \int_0^x [a(s) + b(s)] ds. \quad (44)$$

I give an outline of a proof analogous to that used in Professor Hurwitz's recent article.

Assume

$$\begin{cases} u(x) \equiv U + \left(1 + \frac{a(x)}{\lambda}\right) (\alpha \cos \xi + \beta \sin \xi), \\ v(x) \equiv V + \left(1 + \frac{b(x)}{\lambda}\right) (\beta \cos \xi - \alpha \sin \xi). \end{cases} \quad (45)$$

Putting in (1) we get

$$\begin{cases} N_1(UV\lambda) = \frac{\varphi}{\lambda}, \\ N_2(UV\lambda) = \frac{\psi}{\lambda}, \end{cases} \quad (46)$$

where  $\varphi, \psi, \varphi_\lambda, \psi_\lambda$  are  $O(1)$  as regards  $\lambda$ .

Using as multipliers  $\cos \lambda x, -\sin \lambda x$ , adding, integrating from 0 to  $x$ , and combining this with the result of repeating the procedure with  $\sin \lambda x, \cos \lambda x$  as multipliers we get

$$\begin{cases} U(x) = \frac{F}{\lambda} + \int_0^x [K_{11}U(s) + K_{12}V(s)]ds, \\ V(x) = \frac{G}{\lambda} + \int_0^x [K_{21}U(s) + K_{22}V(s)]ds, \end{cases} \quad (47)$$

where  $F, G$  and the  $K$ 's are  $O(1)$ , and  $F_\lambda, G_\lambda, K_\lambda$  are also.

$$\begin{aligned} \therefore U_\lambda(x) &= \frac{H}{\lambda} + \int_0^x [K_{31}U(s) + K_{32}V(s) + K_{33}U_\lambda(s) + K_{34}V_\lambda(s)]ds, \\ V_\lambda(x) &= \frac{J}{\lambda} + \int_0^x [K_{41}U(s) + K_{42}V(s) + K_{43}U_\lambda(s) + K_{44}V_\lambda(s)]ds, \end{aligned} \quad (48)$$

in which  $H, J$ , and the  $K$ 's of (48) are  $O(1)$ . By the process of successive approximations we obtain from (47), (48),

$$U(x) = O\left(\frac{1}{\lambda}\right), \quad V(x) = O\left(\frac{1}{\lambda}\right), \quad U_\lambda(x) = O\left(\frac{1}{\lambda}\right), \quad V_\lambda(x) = O\left(\frac{1}{\lambda}\right). \quad (49)$$

From (45), (49) we see that (42) is true. Also by differentiating (45) as to  $\lambda$  and using (49) the rest of the proof is obvious.

COROLLARY: For  $|\lambda|$  large  $D(\lambda), D'(\lambda)$  take the forms

$$\begin{cases} D(\lambda) = \sqrt{A^2 + B^2} \sin [\xi(1) + \varphi] + C + O\left(\frac{1}{\lambda}\right), \\ D'(\lambda) = \sqrt{A^2 + B^2} \cos [\xi(1) + \varphi] + O\left(\frac{1}{\lambda}\right), \end{cases} \quad (50)$$

where

$$A \equiv (\alpha\gamma) + (\beta\delta), \quad B \equiv (\alpha\delta) + (\gamma\beta), \quad C = 2(\alpha\beta), \quad (51)$$

and  $\varphi$  is defined by

$$\cos \varphi \equiv \frac{A}{\sqrt{A^2 + B^2}}, \quad \sin \varphi \equiv \frac{B}{\sqrt{A^2 + B^2}}. \quad (52)$$

By Lemma VI, Section I,  $A$  and  $B$  cannot both vanish. From Theorem III

$$\begin{cases} u_1(x) = \cos \xi + O\left(\frac{1}{\lambda}\right), \\ v_1(x) = -\sin \xi + O\left(\frac{1}{\lambda}\right), \\ u_2(x) = \sin \xi + O\left(\frac{1}{\lambda}\right), \\ v_2(x) = \cos \xi + O\left(\frac{1}{\lambda}\right). \end{cases} \quad (53)$$

Putting these in (11) we get

$$D(\lambda) = A \sin \xi(1) + B \cos \xi(1) + (\alpha\beta) + (\gamma\delta) + O\left(\frac{1}{\lambda}\right). \quad (54)$$

Again using the asymptotic values for  $u_{1\lambda}(x)$ ,  $v_{1\lambda}(x)$ , etc., from (53) in the identity

$$D'(\lambda) \equiv \begin{vmatrix} U_1(u_{1\lambda}v_{1\lambda}) & U_1(u_{2\lambda}v_{2\lambda}) \\ U_2(u_{1\lambda}v_{1\lambda}) & U_2(u_{2\lambda}v_{2\lambda}) \end{vmatrix} + \begin{vmatrix} U_1(u_1v_1) & U_1(u_2v_2) \\ U_2(u_{1\lambda}v_{1\lambda}) & U_2(u_{2\lambda}v_{2\lambda}) \end{vmatrix}$$

we derive

$$D'(\lambda) = A \cos \xi(1) - B \sin \xi(1) + O\left(\frac{1}{\lambda}\right),$$

and (50) follows by (3), (51), (52).

To show  $D(\lambda)$  has roots, no matter how large  $|\lambda|$  is, consider the

**THEOREM IV:** *If  $l_n$  is a principal parameter value for the system (1) and the boundary conditions*

$$\begin{cases} \alpha_1 u(0) + \beta_1 v(0) = 0, \\ \gamma_1 u(1) + \delta_1 v(1) = 0, \end{cases} \quad (55)$$

*then there exist exactly two roots of  $D(\lambda)$  in the intervals:*

Case I.  $(l_{2p}, l_{2p+2})$ ,  $p = 0, \pm 1, \pm 2, \dots$ , if  $P_{21} > 0$  for  $\lambda = l_1$ ;

Case II.  $(l_{2p-1}, l_{2p+1})$ ,  $p = 0, \pm 1, \pm 2, \dots$ , if  $P_{21} < 0$  for  $\lambda = l_1$ .\*

Define

$$\begin{cases} B_1(x) \equiv \alpha_1 u(x) + \beta_1 v(x), \\ B_2(x) \equiv \alpha_2 u(x) + \beta_2 v(x), \\ P_1(x) \equiv -\gamma_1 u(x) - \delta_1 v(x), \\ P_2(x) \equiv -\gamma_2 u(x) - \delta_2 v(x), \end{cases} \quad (56)$$

and determine two solutions,  $[u_1(x\lambda), v_1(x\lambda)]$ ,  $[u_2(x\lambda), v_2(x\lambda)]$ , of (1) such that

$$\begin{cases} B_{11}(0) = 0, & B_{21}(0) = 1, \\ B_{12}(0) = 1, & B_{22}(0) = 0; \end{cases} \quad (57)$$

in which the second subscript refers to the solution involved. Without

\* I have followed the method of proof used by Professor Birkhoff in an article published in the *Transactions* in 1909 entitled, "Existence and Oscillation Theorem for a Certain Boundary Value Problem."

loss of generality we may assume that the coefficients  $\alpha_1, \beta_1, \gamma_1, \delta_1$  have been divided by a constant such that

$$(\alpha\beta) = (\gamma\delta) = 1. \quad (58)$$

Then we prove the

LEMMA I: *At a simple value  $\lambda_n$  of  $\lambda$ ,  $D(\lambda)$  changes sign in such a way that  $D'(\lambda)$  has the sign of  $-P_{11}$  or  $P_{22}$ , where*

$$\begin{cases} P_{11} \equiv -\gamma_1 u_1(1) - \delta_1 v_1(1), \\ P_{22} \equiv -\gamma_2 u_2(1) - \delta_2 v_2(1). \end{cases} \quad (59)$$

The solutions  $(u_1, v_1), (u_2, v_2)$  defined by (57) are linearly independent since

$$\begin{vmatrix} u_1(0) & v_1(0) \\ u_2(0) & v_2(0) \end{vmatrix} = \begin{vmatrix} -\beta_1 & \alpha_1 \\ \beta_2 & -\alpha_2 \end{vmatrix} = (\alpha\beta) = -1 \quad (60)$$

and hence by a relation similar to Abel's

$$|u(x), v(x)| = \text{constant} \neq 0.$$

Thus

$$B_{11}(x)B_{22}(x) - B_{12}(x)B_{21}(x) = -1, \quad 0 \leq x \leq 1. \quad (61)$$

Likewise

$$P_{11}(x)P_{22}(x) - P_{12}(x)P_{21}(x) = -1, \quad 0 \leq x \leq 1. \quad (62)$$

Also any solution of (1) may be written

$$\begin{aligned} u(x) &= c_1 u_1(x) + c_2 u_2(x), \\ v(x) &= c_1 v_1(x) + c_2 v_2(x). \end{aligned}$$

If this is to satisfy (2) we must have

$$D(\lambda) = \begin{vmatrix} -P_{11} & 1 - P_{12} \\ 1 - P_{21} & -P_{22} \end{vmatrix} = 0. \quad (63)$$

We have shown that the necessary condition for a double value of  $\lambda$  is that  $D(\lambda)$  be of rank zero, i.e.,

$$\begin{cases} P_{11} = P_{22} = 0, \\ P_{21} = P_{12} = 1. \end{cases} \quad (64)$$

This is obviously also sufficient.

Since

$$N_1(u_1 v_1 \lambda) = N_2(u_1 v_1 \lambda) = 0 \quad (65)$$

and

$$N_1(u_{1\lambda} v_{1\lambda} \lambda) = v_1, \quad N_2(u_{1\lambda} v_{1\lambda} \lambda) = -u, \quad (66)$$

by Lemma II, Section I,

$$u_{1\lambda} v_1 - u_1 v_{1\lambda} = \mathcal{J}_1 \quad (67)$$

Similarly

$$u_{2\lambda} v_2 - u_2 v_{2\lambda} = \mathcal{J}_0^x(u_2^2 + v_2^2) d\lambda. \quad (68)$$

Again from

$$N_1(u_2v_2\lambda) = N_2(u_2v_2\lambda) = 0 \quad (69)$$

and (66) by the same Lemma

$$u_{1\lambda}v_2 - u_2v_{1\lambda} = \int_0^x (u_1u_2 + v_1v_2)ds. \quad (70)$$

Similarly

$$u_{2\lambda}v_1 - u_1v_{2\lambda} = \int_0^x (u_1u_2 + v_1v_2)ds. \quad (71)$$

From (67), (70)

$$u_{1\lambda}(x) = \int_0^x [(u_1^2 + v_1^2)u_2(x) - (u_1u_2 + v_1v_2)u_1(x)]ds. \quad (72)$$

Similarly

$$v_{1\lambda}(x) = \int_0^x [(u_1^2 + v_1^2)v_2(x) - (u_1u_2 + v_1v_2)v_1(x)]ds. \quad (73)$$

Also from (68), (71)

$$u_{2\lambda}(x) = \int_0^x [(u_1u_2 + v_1v_2)u_2(x) - (u_2^2 + v_2^2)u_1(x)]ds, \quad (74)$$

$$v_{2\lambda}(x) = \int_0^x [(u_1u_2 + v_1v_2)v_2(x) - (u_2^2 + v_2^2)v_1(x)]ds. \quad (75)$$

Now from (62), (63)

$$D(\lambda) \equiv P_{12} + P_{21} - 2. \quad (76)$$

Hence by (72), (73), (74), (75), (76)

$$D'(\lambda) = P_{12\lambda} + P_{21\lambda}$$

or

$$D'(\lambda) = \int_0^1 [P_{22}u_1^2 + (P_{12} - P_{21})u_1u_2 - P_{11}u_2^2 + P_{22}v_1^2 + (P_{12} - P_{21})v_1v_2 - P_{11}v_2^2]ds. \quad (77)$$

The integrand consists of two quadratic forms whose discriminant is

$$(P_{12} - P_{21})^2 + 4P_{11}P_{22} \quad (78)$$

or by (62)

$$(P_{12} + P_{21})^2 - 4. \quad (79)$$

This vanishes when  $D(\lambda)$  does. For a simple root of  $D(\lambda)$ ,  $P_{11}$ ,  $P_{22}$  cannot both vanish since then by (78) and (63), (64) would be satisfied. Again neither form can vanish because  $(u_1, v_1)$  and  $(u_2, v_2)$  are linearly independent. Thus the Lemma is proved.

LEMMA II: *At a double value of  $\lambda$ , say  $\lambda_k$ ,  $D(\lambda)$  maintains a negative sign.*

From (64)

$$\Delta P_{11} = \bar{P}_{11}, \quad \Delta P_{22} = \bar{P}_{22}, \quad 1 + \Delta P_{12} = \bar{P}_{12}, \quad 1 + \Delta P_{21} = \bar{P}_{21}, \quad (80)$$

in which

$$\bar{P} \equiv P(\lambda + \Delta\lambda), \quad \lambda = \lambda_k. \quad (81)$$

Then from (62)

$$\bar{P}_{11}\bar{P}_{22} - \bar{P}_{12}\bar{P}_{21} = -1$$

or

$$\Delta P_{11}\Delta P_{22} - (1 + \Delta P_{12})(1 + \Delta P_{21}) = -1, \quad (82)$$

and by (76)

$$\Delta D = \Delta P_{12} + \Delta P_{21} = \Delta P_{11} \Delta P_{22} - \Delta P_{12} \Delta P_{21}. \quad (83)$$

Clearly

$$\Delta P_{11} \equiv -\gamma_1 \Delta u_1(1) - \delta_1 \Delta v_1(1),$$

and so for the others. Also

$$\begin{aligned} N_1(\Delta u, \Delta v, \lambda) &= \Delta \lambda \bar{v}, \\ N_2(\Delta u, \Delta v, \lambda) &= -\Delta \lambda \bar{u}. \end{aligned}$$

Using these for  $(u_1, v_1)$ ,  $(u_2, v_2)$  we get results for  $\frac{\Delta u_1}{\Delta \lambda}$ ,  $\frac{\Delta u_2}{\Delta \lambda}$ ,  $\frac{\Delta v_1}{\Delta \lambda}$ ,  $\frac{\Delta v_2}{\Delta \lambda}$

which differ little from the right members of equations (72) to (75). Putting these values in (83) and omitting in each term infinitesimals of order greater than two, we obtain

$$\Delta D = [\int_0^1 (u_1 u_2 + v_1 v_2) \Delta \lambda ds]^2 - \int_0^1 (u_1^2 + v_1^2) \Delta \lambda ds \int_0^1 (u_2^2 + v_2^2) \Delta \lambda ds.$$

By Lemma IV, Section I,  $\Delta D < 0$ , since the solutions are real, and  $D(\lambda)$  preserves a negative sign at  $\lambda = \lambda_k$ .

LEMMA III:  $D(\lambda)$  has the same sign as  $P_{21}$  at the values  $\lambda = l_n$ ,  $n = 0, \pm 1, \pm 2, \dots$ , unless  $P_{21} = 1$ , when  $D(\lambda) = 0$ .

Since  $u_1(0) = -\beta$ ,  $v_1(0) = \alpha$  (Cf. (60)),  $(u_1, v_1)$  is the only solution except for a constant factor which satisfies (55), (1) for  $\lambda = l_0, l_{\pm 1}, l_{\pm 2}, \dots$  but for no others. (See Professor Hurwitz's article.) These values separate the  $\lambda$ -axis into the intervals

$$\dots (l_{-n}, l_{-n+1}), \dots (l_{-1}, l_0), (l_0, l_1), \dots (l_n, l_{n+1}), \dots \quad (84)$$

By (62) at  $\lambda =$  one of these values, say  $l_r$ ,

$$P_{12} P_{21} = 1. \quad (85)$$

And by (76)

$$D(l_r) = \frac{1}{P_{21}} (1 - P_{21})^2, \quad r = 0, \pm 1, \pm 2, \dots. \quad (86)$$

And the rest of the proof is evident.

Professor Hurwitz has shown that  $P_{11}$  has no double roots, also that for the system (1) and

$$\begin{cases} B_{11}(0) = 0, \\ P_{21}(1) = 0, \end{cases} \quad (87)$$

$P_{21}$  has no double roots. Hence  $P_{11}$ ,  $P_{21}$  change sign when they vanish. Again since

$$P_{11} P_{21\lambda} - P_{11\lambda} P_{21} \equiv u_1 v_{1\lambda} - v_1 u_{1\lambda} \Big|_{x=1},$$

by (67) this cannot vanish but is negative. From these facts it is easy to show that the roots of  $P_{11}$ ,  $P_{21}$  separate each other as well as do those of

$P_{11\lambda}, P_{21\lambda}$ . Clearly then  $P_{21}$  alternates in sign at the values  $\cdots l_{-2}, l_{-1}, l_0, l_1, l_2, \cdots$  and we have two cases:

Case I:  $\left\{ \begin{array}{l} D(\lambda) \geq 0 \text{ at } \lambda = l_{\pm 1}, l_{\pm 3}, \cdots \\ D(\lambda) < 0 \text{ at } \lambda = l_0, l_{\pm 2}, \cdots \end{array} \right\}$  when  $P_{21} > 0$  for  $\lambda = l_1$ ;

Case II:  $\left\{ \begin{array}{l} D(\lambda) < 0 \text{ at } \lambda = l_{\pm 1}, l_{\pm 3}, \cdots \\ D(\lambda) \geq 0 \text{ at } \lambda = l_0, l_{\pm 2}, \cdots \end{array} \right\}$  when  $P_{21} < 0$  for  $\lambda = l_1$ .

There must obviously exist roots of  $D(\lambda)$  as follows:

In Case I—at least two values  $\lambda_n, \lambda_{n+1}$  in each double interval

$$(l_{2p}, l_{2p+2}), \quad p = 0, \pm 1, \pm 2, \cdots,$$

such that

$$l_{2p} < \lambda_n \leq l_{2p+1} \leq \lambda_{n+1} < l_{2p+2} \quad (88)$$

and at least one such that

$$l_1 \leq \lambda_r < l_2.$$

In Case II—at least two values  $\lambda_m, \lambda_{m+1}$  in each double interval

$$(l_{2p-1}, l_{2p+1}), \quad p = 0, \pm 1, \pm 2, \cdots,$$

such that

$$l_{2p-1} < \lambda_m \leq l_{2p} \leq \lambda_{m+1} < l_{2p+1}. \quad (89)$$

To show that there are exactly two roots in  $(l_{2p}, l_{2p+2})$  we make use of the separative property of the roots of  $P_{11}, P_{21}$  and their derivatives. Obviously there must be an even number of roots, since we count a double root as two. If a double root occurs it must fall at  $l_{2p+1}$  by (64) and Lemma III. Then by Lemmas II and I there can be no root elsewhere. If there is no double root, there must be less than four roots, for otherwise there would be at least two in one of the intervals  $(l_{2p}, l_{2p+1}), (l_{2p+1}, l_{2p+2})$ , which violates Lemma I.

Similarly we can show that there is exactly one root in the interval  $(l_1, l_2)$  and exactly two for Case II in the intervals  $(l_{2p-1}, l_{2p+1})$  provided we count a double root at  $l_1$  once each in the intervals  $(l_0, l_1), (l_1, l_2)$ . Thus the Theorem is proved.

COROLLARY. For  $|\lambda|$  large,  $O(1/\lambda_n) = O(1/n\pi)$ ,  $|n|$  large.

Professor Hurwitz has shown that for  $|\lambda|$  large,  $l_n = n\pi + \theta_1 + O(1/n)$ , where  $\theta_1$  is a constant. Thus the intervals  $(l_n, l_{n+2})$  for  $|\lambda|$ ,  $|n|$  large are of length  $2\pi$  to within  $O(1/n)$ . In each of these intervals for  $n$  odd or even according to Case I or II by the Theorem there exist just two values  $\lambda_m, \lambda_{m+1}$  of  $\lambda$  which are roots of  $D(\lambda)$ . Since in any finite interval of the  $\lambda$ -axis there are but a finite number of roots  $l_r$  or  $\lambda_m$ , for  $|n|$  large enough we have

$$\lambda = l_{n+s} - k, \quad 0 \leq |k| \leq \pi,$$

where  $s$  is a finite integer positive or negative, provided we begin to count the  $l$ 's and  $\lambda$ 's so that  $l_0 \geq 0$ ,  $\lambda_0 \geq 0$  and  $l_{-1} < 0$ ,  $\lambda_{-1} < 0$ . Clearly then

$$\lambda_n = n\pi + s\pi - k + \theta_1 + O\left(\frac{1}{n}\right) \quad (90)$$

and since  $n\pi$  is the dominant part,

$$O\left(\frac{1}{\lambda_n}\right) = O\left(\frac{1}{n\pi}\right), \text{ q.e.d.}$$

**THEOREM V:** *If  $(u_n, v_n)$  corresponds to  $\lambda_n$  defined as in the previous Corollary, then this solution of (1), (2) takes the asymptotic form*

$$\begin{cases} u_n(x) = \sin [n\pi x + P(x)] + O\left(\frac{1}{n}\right), \\ v_n(x) = \cos [n\pi x + P(x)] + O\left(\frac{1}{n}\right), \end{cases} \quad (91)$$

when  $|n|$  is large.

By the Corollary of Theorem III  $D(\lambda)$  will have a root when

$$\sin [\xi(1) + \varphi] = -\frac{C}{\sqrt{A^2 + B^2}} + O\left(\frac{1}{\lambda}\right) \quad (92)$$

or when

$$\xi(1) + \varphi = \sin^{-1} \frac{-C}{\sqrt{A^2 + B^2}} + O\left(\frac{1}{\lambda}\right), \quad (93)$$

since by Lemma V, Section I,  $|C| \leq \sqrt{A^2 + B^2}$ , and by Theorem IV the right member of (92) must be numerically  $\leq 1$ , also

$$\sin^{-1} \left[ \frac{-C}{\sqrt{A^2 + B^2}} + O\left(\frac{1}{\lambda}\right) \right] = \sin^{-1} \frac{-C}{\sqrt{A^2 + B^2}} + O\left(\frac{1}{\lambda}\right)$$

and for  $\lambda = \lambda_n$ ,

$$O\left(\frac{1}{\lambda}\right) = O\left(\frac{1}{n\pi}\right) = O\left(\frac{1}{n}\right).$$

Hence from (93), since by (58)  $C > 0$ ,  $D(\lambda) = 0$  when

$$\lambda + \int_0^1 \frac{a(x) + b(x)}{2} dx + \varphi = \sin^{-1} \frac{-C}{\sqrt{A^2 + B^2}} + O\left(\frac{1}{n}\right)$$

and for  $m$  odd

$$\begin{cases} \lambda_n = m\pi + \psi - \int_0^1 \frac{a(x) + b(x)}{2} dx - \varphi + O\left(\frac{1}{n}\right), \\ \lambda_{n+1} = (m+1)\pi - \psi - \int_0^1 \frac{a(x) + b(x)}{2} dx - \varphi + O\left(\frac{1}{n}\right), \end{cases} \quad (94)$$

in which  $\psi$  is the angle in radian measure between 0 and  $\pi/2$  inclusive

whose sine is  $|C|/\sqrt{A^2 + B^2}$ . Clearly (94) may be expressed as

$$\lambda_n = m\pi + (-1)^{m+1}\psi - \int_0^1 \frac{a+b}{2} dx - \varphi + O\left(\frac{1}{n}\right), \text{ all } n, \quad (95)$$

where  $|n|$ ,  $|\lambda_n|$  are large and  $m \equiv n + s$ . From (42)

$$u(x) = \sin(\xi - \theta_0) + O\left(\frac{1}{\lambda}\right),$$

$$v(x) = \cos(\xi - \theta_0) + O\left(\frac{1}{\lambda}\right),$$

provided we choose

$$\alpha \equiv -\sin \theta_0, \quad \beta \equiv \cos \theta_0.$$

This is always possible if we divide  $u(0)$ ,  $v(0)$  by  $\sqrt{[u(0)]^2 + [v(0)]^2}$  and call the quotients  $\alpha$ ,  $\beta$  respectively. Then by (95)

$$u_n(x) = \sin[m\pi x + \theta_2 x + \tfrac{1}{2} \int_0^x [a(s) + b(s)] ds - \theta_0] + O\left(\frac{1}{n}\right),$$

$$v_n(x) = \cos[m\pi x + \theta_2 x + \tfrac{1}{2} \int_0^x [a(s) + b(s)] ds - \theta_0] + O\left(\frac{1}{n}\right),$$

where

$$\theta_2 \equiv (-1)^{m+1}\psi - \int_0^1 \frac{a+b}{2} dx - \varphi$$

and the formulæ (91) follow, provided we define

$$P(x) \equiv \theta_2 x + s\pi x + \int_0^x \frac{a(s) + b(s)}{2} ds - \theta_0,$$

since  $m \equiv n + s$ ,  $\theta_2 \equiv -k + \theta_1$  as shown by (90).

COROLLARY: *If we normalize  $(u_n, v_n)$ , it will have the same form (91).*

#### Section IV. Expansion Theorems.

THEOREM VI: *If two otherwise arbitrary functions  $f(x)$ ,  $g(x)$  satisfy (2) and have continuous second derivatives,  $0 \leq x \leq 1$ , then they are expansible in the form*

$$\begin{aligned} f(x) &\equiv \sum_{n=-\infty}^{+\infty} c_n u_n(x), \\ g(x) &= \sum_{n=-\infty}^{+\infty} c_n v_n(x), \end{aligned} \quad (96)$$

where

$$c_n \equiv \int_0^1 [f(x)u_n(x) + g(x)v_n(x)] dx, \quad (97)$$

and  $(u_n, v_n)$  represents the set of normal orthogonal solutions of (1), (2).

In case  $\lambda_k$  is a double value we count it as two and take any two linearly independent normal orthogonal solutions for that value of  $\lambda$ .

Let us first show that two solutions  $(u_m, v_m)$ ,  $(u_n, v_n)$  of (1), (2) for different values of  $\lambda$  are orthogonal, i.e.,

$$\int_0^1 [u_m(s)u_n(s) + v_m(s)v_n(s)]ds = 0. \quad (98)$$

It is only necessary to apply Lemmas I, III, Section I, and divide by  $\lambda_m - \lambda_n$ . Obviously they are easily normalized by dividing in the case of  $(u_m, v_m)$  by  $\sqrt{\int_0^1 (u_m^2 + v_m^2)dx}$ .

If we assume the series of (96) to be uniformly convergent, then

$$\int_0^1 [f(x)u_m(x) + g(x)v_m(x)]dx = c_m$$

and the series become

$$\begin{cases} \sum_{n=-\infty}^{+\infty} u_n(x) \int_0^1 [f(x)u_n(x) + g(x)v_n(x)]dx, \\ \sum_{n=-\infty}^{+\infty} v_n(x) \int_0^1 [f(x)u_n(x) + g(x)v_n(x)]dx. \end{cases} \quad (99)$$

To study these we assume  $|\lambda_n| > |a(x)|$ , also  $> |b(x)|$ . We have

$$\begin{aligned} u'_n(x) - (\lambda_n + a)v_n(x) &= 0, \\ v'_n(x) + (\lambda_n + b)u_n(x) &= 0. \end{aligned}$$

Then

$$\begin{aligned} \int_0^1 f(x)u_n(x)dx &= - \int_0^1 \frac{f(x)}{\lambda_n + b} v'_n(x)dx \\ &= \left[ -\frac{f(x)v_n(x)}{\lambda_n + b} \right]_0^1 + \int_0^1 v_n(x) \frac{d}{dx} \left[ \frac{f(x)}{\lambda_n + b(x)} \right] dx \\ &= \left[ -\frac{fv_n}{\lambda_n + b} \right]_0^1 + \int_0^1 \frac{u'_n(x)}{\lambda_n + a} \frac{d}{dx} \left[ \frac{f}{\lambda_n + b} \right] dx. \end{aligned}$$

But since by the Corollary of Theorem IV  $O\left(\frac{1}{\lambda_n}\right) = O\left(\frac{1}{n\pi}\right) = O\left(\frac{1}{n}\right)$ , for  $|n|$  large, we have

$$\frac{\frac{d}{dx} \left( \frac{f(x)}{\lambda_n + b(x)} \right)}{\lambda_n + a(x)} = \frac{\frac{f'(x)}{\lambda_n + b} - \frac{f(x)b'(x)}{[\lambda_n + b]^2}}{\lambda_n + a} = O\left(\frac{1}{n^2}\right).$$

$$\therefore \int_0^1 f u_n dx = \left[ -\frac{fv_n}{\lambda_n + b} \right]_0^1 + O\left(\frac{1}{n^2}\right).$$

Similarly

$$\int_0^1 g v_n dx = \left[ \frac{gu_n}{\lambda_n + a} \right]_0^1 + O\left(\frac{1}{n^2}\right).$$

Hence

$$c_n = \left[ \frac{gu_n}{\lambda_n + a} - \frac{fv_n}{\lambda_n + b} \right]_0^1 + O\left(\frac{1}{n^2}\right)$$

or

$$c_n = \frac{1}{\lambda_n} [gu_n - fv_n]_0^1 + O\left(\frac{1}{n^2}\right). \quad (100)$$

Since by hypothesis  $f, g$  satisfy

$$U_1(f, g) = U_2(f, g) = 0, \quad (101)$$

by Lemma III, Section I,  $c_n = O(1/n^2)$  and each series in (99) converges uniformly. If  $f(x), g(x)$  are expansible, we get the series (99) for them. To show that these series converge uniformly to  $f(x), g(x)$  respectively, we define

$$\begin{aligned} F(x) &\equiv \sum_{n=-\infty}^{+\infty} u_n(x) \int_0^1 [f(x)u_n(x) + g(x)v_n(x)]dx, \\ G(x) &\equiv \sum_{n=-\infty}^{+\infty} v_n(x) \int_0^1 [f(x)u_n(x) + g(x)v_n(x)]dx. \end{aligned}$$

Then formally we have

$$\int_0^1 [F(x)u_n(x) + G(x)v_n(x)]dx \equiv \int_0^1 [f(x)u_n(x) + g(x)v_n(x)]dx$$

or

$$\int_0^1 \{[F(x) - f(x)]u_n(x) + [G(x) - g(x)]v_n(x)\}dx = 0, \\ n = 0, \pm 1, \pm 2, \dots$$

If now we define

$$F(x) - f(x) \equiv \psi(x), \quad G(x) - g(x) \equiv -\varphi(x),$$

then by Theorem II we have

$$F(x) \equiv f(x), \quad G(x) \equiv g(x), \quad \text{q.e.d.}$$

**THEOREM VII:** *If  $f(x), g(x)$  do not satisfy (2) but possess continuous second derivatives as before, the series*

$$\begin{aligned} &\{c_0u_0(x) + [c_1u_1(x) + c_{-1}u_{-1}(x)] + [c_2u_2(x) + c_{-2}u_{-2}(x)] + \dots, \\ &\{c_0v_0(x) + [c_1v_1(x) + c_{-1}v_{-1}(x)] + [c_2v_2(x) + c_{-2}v_{-2}(x)] + \dots, \end{aligned} \quad (102)$$

converge uniformly to  $f(x), g(x)$  for  $0 < \epsilon \leq x \leq 1 - \epsilon < 1$ , where  $\epsilon$  is an arbitrarily small positive number and  $(u_n, v_n), c_n$  are defined as before.

We have from (100)

$$c_n u_n(x) = \frac{u_n(x)}{\lambda_n} [g(x)u_n(x) - f(x)v_n(x)]_0^1 + O\left(\frac{1}{n^2}\right).$$

Hence by (91) for  $|n|$  large

$$c_n u_n(x) = \frac{(-1)^n K_1 u_n(x)}{n\pi} - \frac{K_0 u_n(x)}{n^2} + O\left(\frac{1}{n^2}\right),$$

where

$$\begin{aligned} K_1 &\equiv g(1) \sin P(1) - f(1) \cos P(1), \\ K_0 &\equiv g(0) \sin P(0) - f(0) \cos P(0). \end{aligned}$$

Again

$$c_{-n}u_{-n}(x) = \frac{(-1)^n K_1 u_{-n}(x) - K_0 u_{-n}(x)}{-n\pi} + O\left(\frac{1}{n^2}\right).$$

Hence

$$c_n u_n(x) + c_{-n} u_{-n}(x) = \left[ \frac{(-1)^n K_1}{n\pi} - \frac{K_0}{n\pi} \right] [u_n - u_{-n}].$$

But

$$\begin{aligned} u_n(x) - u_{-n}(x) &= \sin [n\pi x + P(x)] - \sin [-n\pi x + P(x)] + O\left(\frac{1}{n}\right) \\ &= 2 \sin (n\pi x) \cos P(x) + O\left(\frac{1}{n}\right). \end{aligned}$$

$$\begin{aligned} \therefore c_n u_n(x) + c_{-n} u_{-n}(x) &= 2 \cos P(x) \left[ K_1 (-1)^n \frac{\sin n\pi x}{n\pi} - K_0 \frac{\sin n\pi x}{n\pi} \right] + O\left(\frac{1}{n^2}\right). \end{aligned}$$

The series  $\sum_{n=1}^{\infty} \frac{(-1)^n \sin n\pi x}{n\pi}$ ,  $\sum_{n=1}^{\infty} \frac{\sin n\pi x}{n\pi}$  converge uniformly, for

$0 < \epsilon \leq x \leq 1 - \epsilon < 1$ , and since we get a similar expression for

$$c_n v_n(x) + c_{-n} v_{-n}(x),$$

the rest of the proof is like in the Theorem above.

## Section V. Further Results on the Distribution of Principal Parameter Values.

Without the use of Theorem IV we can show that for the case in which

$$|C| < \sqrt{A^2 + B^2},$$

$D(\lambda)$  has exactly two roots in each interval

$$[(2k-1)\pi, (2k+1)\pi], \quad |k| \text{ large and integral,}$$

which have the asymptotic form (95) for  $|n|$ ,  $|\lambda|$  sufficiently large. Without loss of generality assume (58), then by definition  $C = 2$ . Then

$$\sqrt{A^2 + B^2} - 2 \equiv E > 0. \quad (103)$$

By (50)

$$D(\lambda) = \sqrt{A^2 + B^2} \sin [\xi(1) + \varphi] + C + \frac{K}{\lambda}, \quad 0 \leq |K| < M,$$

for  $|\lambda|$  large, or

$$\frac{D(\lambda)}{\sqrt{A^2 + B^2}} = \sin \Lambda + \frac{C}{\sqrt{A^2 + B^2}} + \frac{K'}{\lambda},$$

where

$$\Lambda \equiv \xi(1) + \varphi. \quad (104)$$

For  $\Lambda = (2k-1)\pi$  or  $2k\pi$ , and  $|\lambda| > \lambda_M$ ,

$$\frac{D(\lambda)}{\sqrt{A^2 + B^2}} = \frac{2}{\sqrt{A^2 + B^2}} + \frac{K'}{\lambda} > 0.$$

For  $\Lambda = (4k - 1)\pi/2$ , and  $|\lambda| > \lambda_E$ ,

$$\frac{D(\lambda)}{\sqrt{A^2 + B^2}} = -1 + \frac{2}{\sqrt{A^2 + B^2}} + \frac{K'}{\lambda} = \frac{-E}{\sqrt{A^2 + B^2}} + \frac{K'}{\lambda} < 0.$$

Let  $L_1$  be the larger of  $\lambda_M$ ,  $\lambda_E$ , then for  $|\lambda| < L_1$ ,  $D(\lambda)$  must have at least two roots in the intervals  $(2k - 1)\pi < \lambda < 2k\pi$ . Since for  $2k\pi \leq \Lambda \leq (2k + 1)\pi$ ,  $\sin \Lambda \geq 0$ ,  $D(\lambda)$  has no root there. To show that there are exactly two roots in the previous interval we use the fact that  $D(\lambda) = 0$  only when

$$\sin \Lambda = -\frac{2}{\sqrt{A^2 + B^2}} + \frac{K'}{\lambda}.$$

It is obvious that for  $|\lambda| > \text{some } L_2$  the right member is always negative since  $|K'| < M$ . If  $L$  is the larger of  $L_1$ ,  $L_2$ , then, since

$$\sin \Lambda = \frac{E}{\sqrt{A^2 + B^2}} - 1 + \frac{K'}{\lambda},$$

and when  $|\lambda| > \lambda_E$

$$\frac{K'}{\lambda} < \frac{E}{\sqrt{A^2 + B^2}},$$

we have

$$0 > \sin \Lambda > -1, \quad |\lambda| > L.$$

Obviously there are just two values of  $\Lambda$  in any  $[(2k - 1)\pi, 2k\pi]$  interval which satisfy this inequality, i.e.,

$$\Lambda_1 = (2k - 1)\pi + \psi + O\left(\frac{1}{\lambda}\right),$$

$$\Lambda_2 = 2k\pi - \psi + O\left(\frac{1}{\lambda}\right), \quad 0 < \psi < \frac{\pi}{2},$$

where  $\sin \psi \equiv \frac{-2}{\sqrt{A^2 + B^2}}.$

$$\therefore \Lambda = m\pi + (-1)^{m+1}\psi + O\left(\frac{1}{\lambda}\right),$$

for  $|m|$  large, and by the definition of  $\Lambda$

$$\lambda_n = m\pi + (-1)^{m+1}\psi - \int_0^1 \frac{a+b}{2} dx - \varphi + O\left(\frac{1}{m}\right),$$

when  $|n|$  is large; or since in each  $2\pi$ -interval there are 2 values of  $\lambda$  we have

$$\lambda_n = n\pi + s\pi + (-1)^{n+s+1}\psi - \int_0^1 \frac{a+b}{2} dx - \varphi + O\left(\frac{1}{n}\right),$$

$s$  being a finite integer.

CASE of  $C^2 = A^2 + B^2$ : Here the above reasoning does not apply and we attack the problem by getting a more accurate asymptotic expansion

for  $D(\lambda)$  and  $D'(\lambda)$ . If we assume

$$\begin{aligned} u(x) &\equiv e^{\lambda\omega} \left[ u_0 + \frac{u_1}{\lambda} + \frac{u_2}{\lambda^2} + \dots \right], \\ v(x) &\equiv e^{\lambda\omega} \left[ v_0 + \frac{v_1}{\lambda} + \frac{v_2}{\lambda^2} + \dots \right], \end{aligned}$$

then substituting in (1) equate coefficients, we are led to the expansion

$$\begin{aligned} u(x) &\equiv A_0 e^{\xi i} + B_0 e^{-\xi i} + \sum_{k=1}^{\infty} \frac{A_k e^{\xi i} + B_k e^{-\xi i}}{\lambda^k}, \\ v(x) &\equiv C_0 e^{\xi i} + D_0 e^{-\xi i} + \sum_{k=1}^{\infty} \frac{C_k e^{\xi i} + D_k e^{-\xi i}}{\lambda^k}, \end{aligned}$$

or since we restrict  $\lambda$  to real values, to

$$\begin{cases} u(x) \equiv A_0 \cos \xi + B_0 \sin \xi + \sum_{k=1}^{\infty} \frac{A_k \cos \xi + B_k \sin \xi}{\lambda^k}, \\ v(x) \equiv C_0 \cos \xi + D_0 \sin \xi + \sum_{k=1}^{\infty} \frac{C_k \cos \xi + D_k \sin \xi}{\lambda^k}, \end{cases} \quad (105)$$

where  $\xi$  is defined as in (44) and  $A, B, C, D$  are independent of  $\lambda$ .

We may show that  $(\alpha\gamma) = (\beta\delta)$ ,  $(\alpha\delta) = (\gamma\beta)$  so that

$$A = 2(\alpha\gamma), \quad B = 2(\alpha\delta), \quad C = 2(\alpha\beta),$$

and since  $(\alpha\beta) = 1$ , we have

$$\frac{D(\lambda)}{\sqrt{A^2 + B^2}} = 1 + [u_2(1) - v_1(1)] \frac{1}{2} \cos \varphi + [u_1(1) + v_2(1)] \frac{1}{2} \sin \varphi,$$

in which  $(u_1, v_1)$ ,  $(u_2, v_2)$  are defined by (12) and  $\varphi$  by (52). From (50) we can show that  $D'(\lambda)$  has a root in the interval

$$(2k-1)\pi + \frac{1}{4}\pi < \Lambda < (2k-1)\pi + \frac{3}{4}\pi. \quad (106)$$

By putting (105) in (1) and equating coefficients we get a set of equations from which we can solve for  $A, B, C, D$  and if we put that value of  $\lambda$  in for which  $D'(\lambda) = 0$ , we obtain

$$D(\lambda) - \frac{m_1 + 2m_0m_1 \cos 2\varphi + m_0^2}{8\lambda^2} + O\left(\frac{1}{\lambda^3}\right),$$

wherein

$$m_1 \equiv \frac{1}{2}a(1) - \frac{1}{2}b(1), \quad m_0 \equiv \frac{1}{2}a(0) - \frac{1}{2}b(0).$$

Hence  $D(\lambda)$  will have two roots in the interval (106) unless

Case 1)  $m_0 = m_1 = 0$ ;

Case 2)  $\varphi \equiv \cos^{-1}(\alpha\gamma) \equiv \sin^{-1}(\alpha\delta) = \frac{1}{2}\pi$ ,  $m_0 = m_1$ ;

Case 3)  $\varphi = 0$ ,  $m_0 = -m_1$ .

By extending the calculation to the fifth set of functions  $A_4, B_4, C_4, D_4$  we find that  $D(\lambda)$  will still have two roots in (106) for Case 1) unless

$$m_1'^2 + 2m_0'm_1' \cos 2\varphi + m_0'^2 = 0,$$

i.e., unless

$$\begin{aligned} 1^\circ. \quad & m_0' = m_1' = 0; \\ 2^\circ. \quad & \varphi = \tfrac{1}{2}\pi, \quad m_0' = m_1'; \\ 3^\circ. \quad & \varphi = 0, \quad m_0' = -m_1'; \end{aligned}$$

in which

$$m_0' \equiv \tfrac{1}{2}a'(0) - \tfrac{1}{2}b'(0), \quad m_1' \equiv \tfrac{1}{2}a'(1) - \tfrac{1}{2}b'(1).$$

The same is true for Case 2) unless

$$(m_1' - m_0')^2 + 4m_0^2(l_1 - l_0)^2 = 0$$

or

$$\begin{aligned} 1^\circ. \quad & m_1' = m_0', \quad m_0 = 0; \\ 2^\circ. \quad & m_1' = m_0', \quad l_1 = l_0; \end{aligned}$$

where

$$l_1 \equiv \tfrac{1}{2}a(1) + \tfrac{1}{2}b(1), \quad l_0 \equiv \tfrac{1}{2}a(0) + \tfrac{1}{2}b(0).$$

It is probable that by this method  $D(\lambda)$  would also be shown to have roots for Case 3) unless

$$(m_1' + m_0')^2 + 4m_0^2(l_1 + l_0)^2 = 0,$$

and that if one should continue the computation one would find that  $D(\lambda)$  possesses roots unless

$$\begin{aligned} \text{I.} \quad & m_0 = m_1 = m_1' = m_0' = m_1'' = m_0'' \cdots = 0; \\ \text{II.} \quad & \varphi = \tfrac{1}{2}\pi, \quad m_0 = m_1, \quad m_0' = m_1', \quad m_0'' = m_1'', \quad \cdots; \\ \text{III.} \quad & \varphi = 0, \quad m_0 = -m_1, \quad m_0' = -m_1', \quad m_0'' = -m_1'', \quad \cdots. \end{aligned}$$

It is interesting to note that for a restricted case of I, namely

$$a(x) \equiv b(x) \equiv 0,$$

$D(\lambda)$  possesses only double roots, provided

$$\begin{cases} \alpha_1 = \gamma_1 = \beta_2 = \delta_2 = 1, \\ \alpha_2 = \gamma_2 = \beta_1 = \delta_1 = 0. \end{cases} \quad (107)$$

Here

$$\begin{aligned} D(\lambda) &= 2 + 2 \cos \lambda, \\ \lambda &= (2k - 1)\pi, \end{aligned}$$

in fact

$$D(\lambda) \equiv \begin{vmatrix} 1 + \cos \lambda, & \sin \lambda \\ -\sin \lambda, & 1 + \cos \lambda \end{vmatrix}.$$

This leads to a special case of Fourier's Series in which the terms involving  $\sin n\pi x, \cos n\pi x$  for  $n$  even are wanting.

Another interesting result is the following

**THEOREM:** *A necessary condition that the determinant*

$$D(\lambda) \equiv \begin{vmatrix} U_1(u_1v_1) & U_1(u_2v_2) \\ U_2(u_1v_1) & U_2(u_2v_2) \end{vmatrix}$$

*be of rank zero is that  $(\alpha\beta) = (\gamma\delta)$ , where  $(u_1, v_1), (u_2, v_2)$  are defined by (12).*

We have

$$U_1(u_1v_1) \equiv \alpha_1 + \gamma_1u_1(1) + \delta_1v_1(1) = 0, \quad (108)$$

$$U_2(u_1v_1) \equiv \alpha_2 + \gamma_2u_1(1) + \delta_2v_1(1) = 0, \quad (109)$$

$$U_1(u_2v_2) \equiv \beta_1 + \gamma_1u_2(1) + \delta_1v_2(1) = 0, \quad (110)$$

$$U_2(u_2v_2) \equiv \beta_2 + \gamma_2u_2(1) + \delta_2v_2(1) = 0, \quad (111)$$

Multiply the first two equations by  $\delta_2, -\delta_1$  respectively, then

$$(\alpha\delta) + (\gamma\delta)u_1(1) = 0. \quad (112)$$

Similarly using  $-\alpha_2, \alpha_1$  on the last two we have

$$(\alpha\beta) + (\alpha\gamma)u_2(1) + (\alpha\delta)v_2(1) = 0. \quad (113)$$

Again using  $\gamma_2, -\gamma_1$  to multiply the first two, we get

$$(\alpha\gamma) - (\gamma\delta)v_1(1) = 0. \quad (114)$$

Combining equations (112), (114), (113) we obtain

$$(\alpha\beta) - (\gamma\delta) \begin{vmatrix} u_1(1), & v_1(1) \\ u_2(1), & v_2(1) \end{vmatrix} = 0,$$

or

$$(\alpha\beta) = (\gamma\delta). \quad \text{Q.e.d.}$$